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Adjacency of the best and second best valued solutions in combinatorial optimization problems*

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Abstract

We say that a polytope satisfies the strong adjacency property if every best valued extreme point of the polytope is adjacent to some second best valued extreme point for any weight vector. Perfect matching polytopes satisfy this property. In this paper, we give sufficient conditions for a polytope to satisfy the strong adjacency property. From this, binary b-matching polytopes, set partitioning polytopes, set packing polytopes, etc. satisfy the strong adjacency property.

1. Introduction

Given a graph G with edge set E and a weight vector $w \in R^E$, the K-best perfect matching problem finds K different perfect matchings $M_1, \ldots, M_K$ in G such that

$$w(M_1) \le \cdots \le w(M_{K-1}) \le w(M_K) \quad \text{and} \quad w(M_K) \le w(M)$$
$$\text{for all perfect matchings } M \ne M_1, M_2, \ldots, M_K$$

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where $w(M)$ is the sum of the weight of the edges in M. The first research on this problem was done by Murty [4]. Recently Chegireddy and Hamacher [1] developed a different algorithm based on the idea of finding a second best perfect matching M_2 from a given best perfect matching M_1. In [3] this algorithm was modified to efficiently obtain a Kth best valued perfect matching in bipartite graphs. Here a *Kth best valued perfect matching* is a perfect matching M_K such that there exist perfect matchings $M_1, \dots, M_{K-1}$ satisfying

$$w(M_1) < \cdots < w(M_{K-1}) < w(M_K) \quad \text{and} \quad w(M_K) < w(M)$$
$$\text{for all perfect matchings } M \text{ with } w(M) \neq w(M_1), \dots, w(M_K).$$

Note that it is possible to obtain a Kth best valued perfect matching using the algorithms for the K-best perfect matching problem; simply enumerate the perfect matchings in order of nondecreasing cost until one having the Kth best value—i.e., a cost of rank K—is found. However this method is not very efficient since we have output *all* perfect matchings with a cost of rank less than K. Moreover when the problem has a large number of best valued perfect matchings, finding even a second best valued matching may take a lot of time. The modified algorithm in [3] is efficient because it need not find all matchings with cost of rank less than K. In particular, a second best valued matching can be obtained in only one step. This is based on the fact that a second best valued perfect matching can be easily obtained from any best valued perfect matching, which in turn follows from the next well-known lemma.

Lemma 1.1. *For any best valued perfect matching M_1 there exists a second best valued perfect matching M_2 which is adjacent to M_1 on the perfect matching polytope.*

It would be nice to be able to say that this lemma holds for general k, i.e., that there is always a $(k+1)$th best valued perfect matching adjacent to any kth best valued one, but, unfortunately, this is not the case. However we need not despair, although we cannot find a third best valued perfect matching from any second best valued one, we can find a third best valued perfect matching (and a Kth best valued perfect matching), by creating a set of subgraphs of G using the best and second best valued perfect matchings and applying the above lemma to those subgraphs. By using this idea, we posed algorithms in [3] for finding a Kth best valued perfect matching. Since the number of subgraphs on which to apply Lemma 1.1 is not polynomial in K, the worst-case time complexities of our algorithms are not polynomial.

We say that a polytope P satisfies the *strong adjacency property* if every best valued extreme point is adjacent to some second best valued extreme point of P for each weight vector. Lemma 1.1 says that perfect matching polytopes satisfy the strong adjacency property. However polytopes in general, not even those that are 0-1 do not. For example, consider the 0-1 polytope whose extreme points are $\{(0,0,0), (0,1,1), (1,0,1), (1,1,0), (0,0,1)\}$ and the weight vector $w = (-1, -1, -1)$. The best valued extreme points are $(1,1,0), (1,0,1), (0,1,1)$ and the second best valued one is $(0,0,1)$. Clearly $(1,0,1)$ and $(0,1,1)$ are adjacent to $(0,0,1)$, but $(1,1,0)$ is not.

In Section 2 we will give sufficient conditions for a polytope to satisfy the strong adjacency property, and in Section 3 we will discuss some 0-1 polytopes corresponding to combinatorial optimization problems satisfying this property.

2. The strong adjacency property on polytopes

In this section, we pose sufficient conditions for a polytope (not necessarily 0-1) to satisfy the strong adjacency property.

For a polytope P, two vertices $\boldsymbol{v}_1, \boldsymbol{v}_2$ of P are adjacent if there exist no extreme points $\boldsymbol{v}_3, \ldots, \boldsymbol{v}_k$ ($\boldsymbol{v}_3 \neq \boldsymbol{v}_1, \boldsymbol{v}_2$) of P and positive coefficients $\lambda_3, \ldots, \lambda_k$ such that

$$\frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2) = \sum_{i=3}^{k} \lambda_i \boldsymbol{v}_i \quad \text{and} \quad \sum_{i=3}^{k} \lambda_i = 1.$$

Without loss of generality, we may restrict our discussion to full dimensional polytopes because these have at least two extreme points which have distinct objective values for any nonzero weight vector.

We will prove that polytopes having the following property (P1) satisfy the strong adjacency property in Theorem 2.2 below.

(P1) For each ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ of extreme points of P which are not adjacent, there exist other extreme points $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n$ and positive integers $\lambda_1, \ldots, \lambda_n$ such that $\boldsymbol{y} - \boldsymbol{x} = \sum_{i=1}^{n} \lambda_i(\boldsymbol{v}^i - \boldsymbol{x})$.

Combinatorial polytopes [5] are a special class of polytopes satisfying property (P1). A *combinatorial polytope* is a 0-1 polytope which satisfies the following: if two extreme points $\boldsymbol{x}$ and $\boldsymbol{y}$ are not adjacent then there exist two other exteme points $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ such that $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{v}^1 + \boldsymbol{v}^2$. It is easy to see that they are polytopes satisfying (P1) with $n = 2$ and $\lambda_1 = \lambda_2 = 1$. Note that there are 0-1 polytopes satisfying (P1) but which are not combinatorial. For instance, the polytope whose extreme points are $\{(0000), (1111), (1000), (0001), (0110), (1101), (1011)\}$ satisfies (P1) but is not combinatorial, since all pairs of extreme points of P are adjacent except the pair (0000) and (1111).

If a polytope satisfies (P1) then it satisfies a property slightly stronger than (P1).

Lemma 2.1. *Let P be a polytope having property* (P1). *Then for each ordered pair $(\boldsymbol{x}, \boldsymbol{y})$ of extreme points which are not adjacent, there exist extreme points $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^n$ adjacent to $\boldsymbol{x}$ and positive integers $\lambda_1, \ldots, \lambda_n$ such that $\boldsymbol{y} - \boldsymbol{x} = \sum_{i=1}^{n} \lambda_i(\boldsymbol{v}^i - \boldsymbol{x})$.*

Proof. Assume on the contrary that this does not hold for some ordered pair $(\boldsymbol{x}, \boldsymbol{y})$. Since P satisfies (P1), $\boldsymbol{y} - \boldsymbol{x}$ can be expressed as

$$\boldsymbol{y} - \boldsymbol{x} = \sum_{i=1}^{m} \mu_i(\boldsymbol{u}^i - \boldsymbol{x}) \tag{2.1}$$

for some extreme points $u^1, \ldots, u^m$ ($\neq x, y$) and some positive integers $\mu_1, \ldots, \mu_m$. On the other hand, there is a vector c such that

$$c^{\mathrm{T}}(z - x) > 0 \quad \text{for all } z \in P - \{x\}, \tag{2.2}$$

because x is an extreme point of P. Without loss of generality, we can assume that y minimizes $c^{\mathrm{T}}(y - x)$ among counterexamples for fixed x, i.e.,

$$\begin{aligned} &c^{\mathrm{T}}(y - x) \leq c^{\mathrm{T}}(y' - x) \quad \text{for all } y': \\ &\quad (x, y') \text{ does not satisfy the lemma.} \end{aligned} \tag{2.3}$$

Then the following inequalities follow from (2.1) and (2.2)

$$c^{\mathrm{T}}(y - x) > c^{\mathrm{T}}(u^i - x) \quad \text{for } i = 1, \ldots, m. \tag{2.4}$$

From (2.3) and (2.4), if x and u^i are not adjacent then

$$u^i - x = \sum_{j=1}^{m_i} \mu_{i_j}(u^{i_j} - x) \tag{2.5}$$

holds for some extreme points $u^{i_1}, \ldots, u^{i_{m_i}}$ adjacent to x and some positive integers $\mu_{i_1}, \ldots, \mu_{i_{m_i}}$. From (2.1) and (2.5), $y - x$ can be expressed as a positive integer combination of vectors $v^i - x$ where v^i and x are adjacent, a contradiction. Hence this lemma holds. □

Theorem 2.2. *If a polytope P satisfies condition* (P1) *then P satisfies the strong adjacency property.*

Proof. Let w be any weight vector and x be a minimum extreme point for w. Suppose that y is a second best valued extreme point. If x and y are adjacent, there is nothing to prove. Assume that x and y are not adjacent. From Lemma 2.1, $y - x = \sum_{i=1}^{n} \lambda_i(v^i - x)$ holds for some extreme points $v^1, \ldots, v^n$ which are adjacent to x and some positive integers $\lambda_1, \ldots, \lambda_n$. Since x is a minimum extreme point and y is a second minimum one for w, either $w^{\mathrm{T}}(v^i - x) = 0$ or $w^{\mathrm{T}}(y - x) \leq w^{\mathrm{T}}(v^i - x)$ holds for each i. Then there is exactly one second best valued extreme point v^j in $\{v^1, \ldots, v^n\}$ and the others are minimum extreme points because every coefficient λ_i is a positive integer. Hence there exists a second best valued extreme point v^j adjacent to x. □

Corollary 2.3. *Combinatorial polytopes satisfy the strong adjacency property.*

Note that condition (P1) is not necessary for a polytope to satisfy the strong adjacency property even when we deal only with 0-1 polytopes. For example, consider the polytope whose extreme points are {(0000), (1111), (1110), (1101), (1011), (0111), (1001), (0101), (0011)}. All ordered pairs of extreme points which are not adjacent, except the ordered pair ((0000), (1111)), satisfy the condition in property (P1). From the proof of Theorem 2.2, for any weight vector, every best valued extreme point except (0000) is adjacent to some second best valued extreme point. On the other hand, it is not difficult to show that (0000) is adjacent to some second best valued extreme point for any weight vector for which (0000) is minimum and (1111) is second best valued. Hence, the polytope satisfies the strong adjacency property.

Lemma 2.4. *If P satisfies the strong adjacency property then any face F of P does also.*

Proof. Let w be a weight vector and let x be a minimum extreme point of F for w. Since F is a face of P, there is a vector c such that

$$c^{\mathrm{T}}(z-y)>0 \quad \text{and} \quad c^{\mathrm{T}}(x-y)=0$$
$$\text{for all } z \in P-F \text{ and for all } y \in F. \tag{2.6}$$

For sufficiently small positive number ε, x is a minimum extreme point of P with respect to $c+\varepsilon w$. Then there is a second best valued extreme point y of P adjacent to x with respect to $c+\varepsilon w$. Since ε is sufficiently small, y belongs to F. From (2.6), y is a second best valued extreme point of F for w. □

3. Applications to combinatorial optimization problems

Here we discuss some examples of 0-1 integer programming problems whose feasible regions have the strong adjacency property. The results below have already been proved, however, here we give other simple proofs using Theorem 2.2.

Corollary 3.1 [6]. *Let A be an $(m \times E)$-matrix and b an m-dimensional vector. Then the convex hull P of $\{x \in \{0,1\}^E \mid Ax=b\}$ satisfies the strong adjacency property.*

Proof. Let x^1, x^2 be two nonadjacent extreme points of P. Then there exist extreme points $x^3, \ldots, x^k$ and positive coefficients $\lambda_3, \ldots, \lambda_k$ such that $\frac{1}{2}(x^1+x^2)=\sum_{i=3}^{k}\lambda_i x^i$, $\sum_{i=3}^{k}\lambda_i=1$ and x^3 is different from x^1 and x^2. Let $y=x^1+x^2-x^3$ and consider y_j. The positivity of λ_3 implies that if $x_j^1=x_j^2=0$ then $x_j^3=0$. Conversely if $x_j^3=1$ then at least one of x_j^1 and x_j^2 must be equal to 1. Finally, if $x_j^1=x_j^2=1$ then x_j^3 must also be 1, since if $x_j^3=0$ then $1=\frac{1}{2}(x_j^1+x_j^2)=\sum_{i=3}^{k}\lambda_i x_j^i=\sum_{i=4}^{k}\lambda_i x_j^i \le \sum_{i=4}^{k}\lambda_i<1$. Therefore $y=x^1+x^2-x^3$ is a 0-1 vector. On the other hand, $Ay=Ax^1+Ax^2-Ax^3=b$. Thus y is an extreme point of P because P is a 0-1 polytope. Since x^3 is different from x^1 and x^2, y is also different from x^1 and x^2. Furthermore, if $y=x^3$ then $x^1=x^2$, contrary to the assumption that x^1 and x^2 are nonadjacent extreme points of P. Hence P is combinatorial, i.e., it satisfies the strong adjacency property. □

Clearly, perfect matching polytopes (on general graphs) are special cases of the above corollary. Moreover, Corollary 3.1 shows that binary b-matching polytopes and set partitioning polytopes satisfy the strong adjacency property. We can prove the following easily in the same way as Corollary 3.1.

Corollary 3.2. *Let A be an $(m \times E)$-matrix with entries* 0 *or* 1, *and* **1** *the m-dimensional vector with all entries equal to* 1. *Then the convex hull of* $\{x \in \{0, 1\}^E \mid Ax \leq \mathbf{1}\}$ *satisfies the strong adjacency property.*

Proof. Since the entries of matrix A are 0 or 1, the slack of each inequality is also 0 or 1. So by adding the slack variables, we can prove this corollary in the same way as Corollary 3.1. □

Corollary 3.2 implies that matching polytopes (on general graphs), stable set polytopes and set packing polytopes satisfy the strong adjacency property.

Finally, we consider base polytopes and independence polytopes of matroids. We call the convex hull of bases (or independent sets) of a matroid the *base polytope* (or *independence polytope*) of the matroid.

It is known that the base polytope of a matroid is a combinatorial polytope [5].

Corollary 3.3 [2, 5]. *Base polytopes of matroids satisfy the strong adjacency property.*

Corollary 3.4 [2]. *Independence polytopes of matroids satisfy the strong adjacency property.*

By applying the strong adjacency property to assignment problems, a practically efficient algorithm for finding a Kth best valued perfect matching in bipartite graphs was constructed in [3]. The time complexity of the algorithm is not polynomial in terms of K but is polynomial when K is a fixed number. This indicates that there may be applications of the strong adjacency property to construct other such algorithms to the above polytopes.

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